LIMITING SELF-SIMILAR, ONE-DIMENSIONAL, NON-STEADY MOTIONS OF A GAS (CAUCHY'S PROBLEM AND THE PISTON PROBLEM)

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The work of Barenblatt [1] deals with certain problems about non-steady one-dimensional filtration of liquid in a porous medium and about fluid motion in a boundary layer, which can be obtained from corresponding similarity problems by passage to a limit. Because of this, these motions are called limiting self-similar motions.

Considered below are limiting self-similar motions of an ideal, non-heat-conducting, perfect gas in cases of Cauchy's problem for the equations of one-dimensional, non-steady motion with plane waves (the one-dimensional similarity problem of Cauchy allows the above mentioned limiting transition only in the case of motion with plane waves) and the problem of a symmetrical piston as it displaces a gas.

The existence of a solution for the system of equations for one-dimensional motions of gas with plane waves which have the form of limiting, similarity solutions, was indicated by Staniukovich [2] although he did not give an actual construction of a solution of a concrete problem. Cauchy's similarity problem for one-dimensional non-steady motions of gas was considered by us in Ref. [7], where it was established that this problem does not always have a solution.

In the present work, in the investigation of limiting similarity Cauchy problems, we shall prove that the same phenomenon occurs here also, and that more than one solution exists for certain conditions of the considered problem.

In Ref. [7] we studied the self-similar motions of a gas displaced by a symmetrical piston, considered previously by Sedov [3, 4].

Krashennikova [5] and Chernyi [6] established that this problem also does not always have a solution. In the present investigation of limiting similarity piston problems, we shall establish that this problem always has a single-valued solution, which was to be expected on the basis of the results of Ref. [7].

In this work we make only a qualitative investigation concerning the principles of the presented problems, and no numerical results of solutions are given, although they are easily obtainable when necessary.

1. Cauchy's problem.

(1) Let us consider the following problem of Cauchy.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \qquad \frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \qquad \frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+u\frac{\partial}{\partial x}\left(\frac{p}{\rho^{\gamma}}\right)=0 \quad (1.1)$$

$$p(x,0)=Ax^{\alpha}, \quad \rho(x,0)=Bx^{\beta}, \quad u(x,0)=M\sqrt{\frac{A}{B}}\,x^{\frac{\alpha-\beta}{2}} \quad (x>0) \quad (1.2)$$

$$p(x,0)=LA(-x)^{\alpha}, \quad \rho(x,0)=NB(-x)^{\beta}, \quad u(x,0)=M_1\sqrt{\frac{A}{B}}(-x)^{\frac{\alpha-\beta}{2}} \quad (x<0) \quad (1.3)$$

The solution of this problem is self-similar and has the following form [3, 4, 7]

$$u(x,t)=\frac{x}{t}V(\lambda), \quad p(x,t)=B|x|^{\beta+2}t^{-2}P(\lambda), \quad \rho(x,t)=B|x|^{\beta}R(\lambda) \quad (1.4)$$

where

$$\lambda=\mu\sqrt{\frac{A}{B}}\,t\,|x|^{\frac{\alpha-\beta}{2}-1} \quad (\mu=\text{const}) \quad (1.5)$$

and the functions V, R, $P = RZ/\gamma$ are determined by the ordinary differentia equations,

$$\frac{dZ}{dV}=Z\frac{[2(V-1)+(\gamma-1)V](V-q)^2-(\gamma-1)V(V-1)(V-q)-[2(V-1)+\varkappa(\gamma-1)]Z}{(V-q)[V(V-1)(V-q)+(\varkappa-V)Z]} \quad (1.6)$$

$$\frac{d\ln\lambda}{dV}=\frac{1}{q}\frac{(V-q)^2-Z}{V(V-1)(V-q)+(\varkappa-V)Z} \quad (1.7)$$

$$\frac{V-q}{q}\frac{d\ln R}{d\ln\lambda}=\frac{V(V-1)(V-q)+(\varkappa-V)Z}{Z-(V-q)^2}+(\beta+1)V \quad (1.8)$$

$$q=2/[2-(\alpha-\beta)], \quad \varkappa=-\alpha q/\gamma \quad (1.9)$$

Let us replace x by $x + a\alpha$ everywhere and let $\mu = \alpha$, $A = p_0 (a\alpha)^{-\alpha}$,

$$B = \rho_0(a\alpha)^{-\beta},\ a\alpha = b\beta,\ V = (1/\alpha) V_1,\ P = (1/\alpha^2)P_1,\ R = R_1$$

and pass to the limit for $\alpha \to \infty$. Equations (1.2) will then yield

$$p(x,\ 0) = p_0 e^{x/a}, \qquad \rho(x,\ 0) = \rho_0 e^{x/b}, \qquad u(x,\ 0) = M \sqrt{\frac{p_0}{\rho_0}}\, e^{\frac{b-a}{2ab}x} \tag{1.10}$$

Equations (1.3) need not be considered, as the region determined by them goes to the left, into infinity. Equations (1.4) and (1.5) will transform as follows: (indices omitted)

$$u(x,\ t) = \frac{a}{t} V(\lambda), \qquad p(x,\ t) = \rho_0 \frac{a^2}{t^2}\, e^{x/b} P(\lambda), \qquad \rho(x,\ t) = \rho_0 e^{x/b} R(\lambda) \tag{1.11}$$

$$\lambda = \sqrt{\frac{p_0}{\rho_0}} \frac{t}{a}\, e^{\frac{b-a}{2ab}x} \tag{1.12}$$

The differential equations (1.6), (1.7), (1.8), will transform into the following:

$$\frac{dZ}{dV} = Z\, \frac{\left(2 - \frac{\gamma - 1}{\gamma} n\right) Z + (\gamma - 1) V (V + n) - 2 (V + n)^2}{(V + n) \left[\frac{n}{\gamma} Z - V (V + n)\right]} \tag{1.13}$$

$$\frac{d \ln \lambda}{dV} = \frac{Z - (V + n)^2}{n \left[\frac{n}{\gamma} Z - V (V + n)\right]} \tag{1.14}$$

$$\frac{V + n}{n} \frac{d \ln R}{d \ln \lambda} = - \frac{\frac{n}{\gamma} Z - V(V + n)}{Z - (V + n)^2} - \left(1 - \frac{2}{n}\right) V \tag{1.15}$$

where $n = 2b/(b - a)$.

(2) For the consideration of discontinuous solutions we shall need the conditions at a shock wave, in non-dimensional form; these are given by equations (2.7) and those preceeding it in Chapter IV of Ref. [4] from these formulas at the limit $a \to \infty$ we will obtain

$$V_2 + n = (V_1 + n)\left[1 + \frac{2}{\gamma + 1} \frac{Z_1 - (V_1 + n)^2}{(V_1 + n)^2}\right], \qquad R_2 = R_1 \frac{V_1 + n}{V_2 + n} \tag{2.1}$$

$$Z_2 = \left(\frac{\gamma - 1}{\gamma + 1}\right)^2 \frac{1}{(V_1 + n)^2} \left[(V_1 + n)^2 + \frac{2}{\gamma - 1} Z_1\right] \left[\frac{2\gamma}{\gamma - 1} (V_1 + n)^2 - Z_1\right]$$

Relations (2.1) transform the points of the upper half-plane of the plane V, Z in the following way (points $Z < 0$ do not have any physical meaning). The points of the parabola (Fig. 1)

$$Z = (V + n)^2 \tag{2.2}$$

transform into themselves; the points on the axis $z = 0$ transform into points of the parabola

$$Z = \frac{2\gamma}{\gamma - 1}(V + n)^2 \tag{2.3}$$

The points of the region between the axis $Z = 0$ and the parabola (2.2) transform into points of the region between parabolas (2.2) and (2.3). Here the first points represent the conditions of the gas in front of the shock wave, the second points represent the conditions behind it. It is impossible to be in the region bounded on the bottom by parabola (2.3).

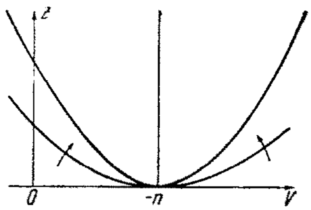

Fig. 1.

(3) The conclusion that the required limiting solutions have the form (1.11) and (1.12) can be also reached by the following method. We are looking for a solution to Cauchy's problem for the system (1.1) and the initial conditions:

$$p(x, 0) = p_0 e^{x/a}, \quad \rho(x, 0) = \rho_0 e^{x/b}$$
$$u(x, 0) = U e^{x/c} \tag{3.1}$$

The solution depends upon the system of determining parameters, x, t, p_0, ρ_0, U, a, b, c, γ from which the following independent non-dimensional combinations can be made up:

$$\lambda = \sqrt{\frac{p_0}{\rho_0}}\frac{t}{a}, \quad \xi = \frac{x}{a}, \quad \pi_1 = U\sqrt{\frac{\rho_0}{p_0}}, \quad \pi_2 = \frac{b}{a}, \quad \pi_3 = \frac{c}{a}, \gamma$$

It will have the form [4]

$$p = p_0 P(\lambda, \xi, \pi_i, \gamma), \ \rho = \rho_0 R(\lambda, \xi, \pi_i, \gamma), \ u = \sqrt{\frac{p_0}{\rho_0}} V(\lambda, \xi, \pi_i, \gamma) \tag{3.2}$$

Let us make the transformation $x = x' + x_0$. The system (1.1) is invariant with respect to this transformation and the initial conditions are changed as follows

$$p(x', 0) = p_0' e^{x'/a}, \quad \rho(x', 0) = \rho_0' e^{x'/b}, \quad u(x', 0) = U' e^{x'/c} \tag{3.3}$$

where $p_0' = p_0 e^{x_0/a}$, $\rho_0' = \rho_0 e^{x_0/b}$, $U' = U e^{x_0/c}$, i.e. they also keep their form.

In this manner, Cauchy's problem (1.1), (3.3) differs from the similar problem (1.1), (3.1) only in the values of p_0, ρ_0 and U. Hence, the solution of problem (1.1), (3.3) will be obtained from the solution of problem (1.1), (3.1) i.e. from (3.2) if, in the latter, we replace

p_0, ρ_0 and U by p_0', ρ_0', U' and x by x' i.e.

$$p(x',\ t) = p_0 e^{\frac{x_0}{a}} P\left(\sqrt{\frac{p_0}{\rho_0}}\frac{t}{a}\ e^{\frac{b-a}{2ab}x_0},\ \frac{x'}{a},\ U\sqrt{\frac{\rho_0}{p_0}}e^{\left(\frac{1}{c}-\frac{b-a}{2ab}\right)x_0},\ \frac{b}{a},\ \frac{c}{a},\ \gamma\right)$$

$$\rho(x',\ t) = \rho_0 e^{\frac{x_0}{b}} R\left(\sqrt{\frac{p_0}{\rho_0}}\frac{t}{a}e^{\frac{b-a}{2ab}x_0},\ \frac{x'}{a},\ U\sqrt{\frac{\rho_0}{p_0}}e^{\left(\frac{1}{c}-\frac{b-a}{2ab}\right)x_0},\ \frac{b}{a},\ \frac{c}{a},\ \gamma\right) \quad (3.4)$$

$$u(x',\ t) = Ue^{\frac{x_0}{c}} V\left(\sqrt{\frac{p_0}{\rho_0}}\frac{t}{a}\,e^{\frac{b-a}{2ab}x_0},\ \frac{x'}{a},\ U\sqrt{\frac{\rho_0}{p_0}}e^{\left(\frac{1}{c}-\frac{b-a}{2ab}\right)x_0},\ \frac{b}{a},\ \frac{c}{a},\ \gamma\right)$$

But for $x' = x - x_0$, (3.4) has to coincide with (3.2); the relations obtained by equating the right-hand sides of (3.4) and (3.2) for $x' = x - x_0$ are identical with respect to x_0. Putting $x_0 = x$ in these identities and requiring that a, b and c be related by

$$\frac{1}{c} - \frac{b-a}{2ab} = 0 \quad (3.5)$$

we will obtain

$$P\left(\sqrt{\frac{p_0}{\rho_0}}\frac{t}{a},\ \frac{x}{a},\ U\sqrt{\frac{\rho_0}{p_0}},\ \frac{b}{a},\ \frac{c}{a},\ \gamma\right) = $$
$$= e^{\frac{x}{a}} P\left(\sqrt{\frac{p_0}{\rho_0}}\frac{t}{a}e^{\frac{b-a}{2ab}x},\ 0,\ U\sqrt{\frac{\rho_0}{p_0}},\ \frac{b}{a},\ \frac{c}{a},\ \gamma\right) \quad (3.6)$$

and analogous relations for R and V.

In this manner, we can see that by satisfying the conditions (3.5) the solution of Cauchy's problem (1.1), (3.1) has the "self-similar" form (3.7) i.e. it is determined by ordinary differential equations. Obvious transformations will reduce these equations to the system (1.13), (1.14) and (1.15).

(4) According to the theory of Kovalevskaya, the problem (1.1), (1.10) has a unique analytic solution in the vicinity of any finite point of the axis $t = 0$. Therefore system (1.13), (1.14) and (1.15) near $\lambda = 0$ has a unique analytical solution in terms of λ which is determined by the initial conditions

$$V(0) = P(0) = 0,\ R(0) = 1,$$

Thus we have the following expansions

$$V = M\lambda + \ldots,\ R = 1 + \ldots,\ \ P = \lambda^2 + \ldots,$$

from which it follows that the required integral curve of equation (1.13) near the point $V = Z = 0$ has the following representation

$$Z = (\gamma/M^2)V^2 + \ldots \quad \text{for } M \neq 0, \qquad Z = -\gamma V + \ldots \quad \text{for } M = 0.$$

The last equations show that on the plane V, Z, the origin of coordinates is also the nodal point of the integral curves of equation (1.13), where the different curves of this nodal point correspond to the various distributions of initial velocities. To solve Cauchy's problem means to complete all the integral curves, issuing from the origin of the coordinates, up to the points where $\lambda = \infty$ or to points located on the straight line $V = -n$, which corresponds to the limit of the region occupied by the moving gas (piston, vacuum). With this, the parameter λ has to vary along the integral curves monotonically. On the parabola (2.2) λ reaches a stationary value, therefore a continuous transition across it along the integral curves is inadmissible [4, 7].

Equations (1.13), (1.14) and (1.15) contain only one parameter n, therefore the study of the class of Cauchy's problems encountered here is limited to the study of the field of integral curves of equation (1.13) and the distribution of λ and R along them for all possible values of n.

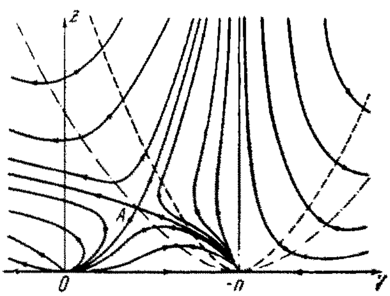

Fig. 2.

The field of integral curves for $n < 0$ is represented in Fig. 2. Arrows on the integral curves represent the direction of increase of the parameter λ. The dotted curves are the parabolas (2.2), (2.3). At point A having the coordinates $V = n^2/(\gamma - n)$, $Z = [\gamma n/(\gamma - n)]^2$, we have a saddle point.

When moving along the separatrix to point A we have asymptotically

$$\lambda = \lambda_0 \exp c\left(V - \frac{n^2}{\gamma - n}\right) \qquad c = \frac{\gamma}{n^2}\,\frac{a - 2\gamma\dfrac{n}{\gamma - n}}{a - \gamma\dfrac{\gamma + n}{\gamma - n}}$$

a is the tangent of the angle of inclination of the separatrix at the point A, with $a \neq 2\gamma n/(\gamma - n)$, $a \neq \gamma(\gamma + n)/(\gamma - n)$. Therefore λ at point A is finite and a transition can be made through point A by the integral curves (separatrices). With this, λ at point A will vary monotonically. With $\gamma < 3$ and $|V| \to \infty$ we will have

$$Z = a_1 |V|^{3-\gamma} + \ldots, \; \lambda = c_1 \exp(V/n) + \ldots \tag{4.1}$$

where a_1 and c_1 are constants. All this results in the following.

For integral curves emanating from the origin to the left of the separatrix that enters saddle A, the solution of Cauchy's problem is continuous and determined for all values of x and $t > 0$. For the remaining group of integral curves, the solution can be constructed in the following manner. From these curves the mapping point makes a jump onto the separatrix that goes through the saddle A and the point $Z = 0$, $V = -n$, and continuing along this separatrix goes through A and thence to infinity at $V \to -\infty$. The solution to Cauchy's problem is discontinuous and the motion takes place with one shock wave.

For the proof of these statements let us note the following. As a part of the separatrix, (through the points A and $Z = 0$, $V = -n$) is located between parabolas (2.2) and (2.3) and crosses the second parabola at $Z > 0$, then the image of this piece obtained by means of mapping (2.1) appears as some continuous curve without self-intersection that lies between parabola (2.2) and the axis $Z = 0$, and connects point A with some point of the $Z = 0$ axis for which $V < -n$. We shall show that this image is contained wholly in the region bounded by the segment $Z = 0$, $0 < V < -n$, a section of the separatrix OA, and the section of the parabola (2.2) that joins point A and $Z = 0$, $V = -n$. The image of the straight line $V = V_A = h^2/(\gamma - n)$ according to (2.1) is a parabola, that goes through the points A and $Z = 0$, $V = -n$. It is easy to show that the portion of this parabola on the interval $V_A < V < -n$ is located above the part of the separatrix that connects the points A and $Z = 0$, $V = -n$. Therefore the image of this piece is located to the right of the vertical line $V = V_A$, which proves our previous statement. Thus, this image crosses every integral curve that emanates from point O to the left of the separatrix OA, and does not cross any other curve emanating from point O. This proves the existence of a solution to Cauchy's problem.

For the proof of uniqueness we may note (and it can be easily shown) that through the points A and $Z = 0$, $V = -n$ we can draw two parabolas that have a vertical axis and are tangent from opposite sides to the piece of the separatrix which connects the two points, so that this piece is contained between the parabolas. But, as can be proven, the images of those parabolas are also parabolas with a vertical axis, and they cross the axis $Z = 0$ to the right of $V = V_A$. Between these images the image of the piece of the separatrix is included. Therefore, this image, which at least near the point A is a monotonically descending curve with a negative slope, will intersect every integral curve only once, as these curves in the considered region have a positive slope. It seems unrealistic to suppose that in going away from point A this slope might become so small that some integral curves would be intersected more than once, so that uniqueness of solution of Cauchy's problem would not exist for some values of M. In fact, this image is contained in the narrow strip between the two parabolas, the slope of which is negative, and it is hard to assume that the curvature of the image can change so drastically as to allow the possibility of more than one solution. A rigorous proof for this is apparently impossible.

The uniqueness of the solution for Cauchy's problem could be disrupted if we could make a jump from integral curves emanating from the point O onto curves going to the point $V = -n$, $Z = \infty$, at which point we would have $P = 0$ (empty space). When a jump is made from a certain selected curve onto a certain curve as just indicated, we will reach the point $V = -n$, $Z = \infty$ with a completely determined value $\lambda = \lambda^*$. A jump can be made from a given curve onto different curves going to $V = -n$, $Z = \infty$, obtaining different values of λ^*. Solutions corresponding to such discontinuous integral curves describe motion in which a vacuum is formed. The motion of the boundary is determined by the equation $\lambda = \lambda^*$, i.e.

$$x^* = -\frac{2ab}{b-a}\left[\ln \sqrt{\frac{p_0}{\rho_0}}\frac{t}{a} - \ln \lambda^*\right] \tag{4.2}$$

From the asymptotic formulas

$$Z \approx C(V+n)^{\frac{2\gamma}{n}-(\gamma-1)}, \qquad R \approx C_1(V+n)^{-\frac{2\gamma}{n}+\gamma-1}$$

$$P \approx \frac{1}{\gamma}CC_1, \qquad \lambda \approx \lambda^* \exp\left[\frac{\gamma}{n^2}(V+n)\right] \tag{4.3}$$

which are valid in the vicinity of $V = -n$, $Z = \infty$, it follows that the pressure on the boundary of the created vacuum is different from 0, i.e. a cavity is formed because the gas is displaced by a piston moving from infinity, according to the law (4.2). Therefore, the above solution corresponds to the motions of the gas, which are due to the initial non-equilibrium distributions (1.10) and the displacing action of the piston. The above mentioned possibility of making a jump from a given curve emanating from the origin of the coordinates onto various curves going to the point $V = -n$, $Z = \infty$, obtaining in this manner different values for λ^*, corresponds to motions with identical initial states, but with different motions of the piston (4.2).

From the above statements it follows that motion without a piston, i.e. Cauchy's problem, exists corresponding either to continuous integral curves, mentioned above, or discontinuous curves composed of parts of integral curves emanating from the origin of the coordinates to the right of the separatrix and a part of the other separatrix. It can be shown that the above discussions are valid for $n < 0$ for all values of $\gamma > 1$. Therefore the solution of Cauchy's problem for the case $n < 0$ exists. It is unique and determined for all values of x and $t > 0$; the solution is continuous for $M \leqslant M_0$ and discontinuous for $M > M_0$, where M_0 is the value of M corresponding to the separatrix that goes through the origin of the coordinates.

In Fig. 3 we have a representation of the field of the integral curves for the case $0 < n < \gamma < 3$. A study of this field shows that if the image of point 0 is located on the parabola (2.3) above the point of

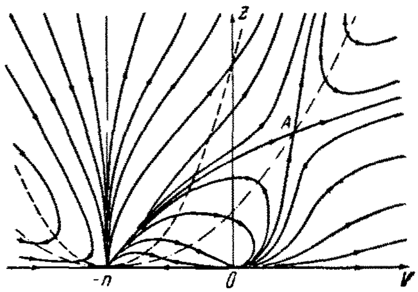

Fig. 3.

intersection of this parabola with the separatrix that goes through points $Z = 0$, $V = -n$ and point A, the solution of Cauchy's problem for all values of M exists, but it is not unique. Here for each value of M there is an infinite number of possible solutions which, however, will coincide for small values of λ. If, however, the indicated points are distributed on the parabola (2.3) in an opposite order, then there exists a value of $M = M_0$ such that, for $M < M_0$, the solution of Cauchy's problem cannot be continued for all values of $t > 0$. For $M > M_0$ this continuation can be achieved in infinitely many ways, i.e. the solution is not unique. For $M = M_0$ the solution is continuous and unique. Both of these possibilities actually occur, since for $\gamma = 2$ the equation of the separatrix that goes through the points $Z = 0$, $V = -n$ and A, is $Z = -[2n/(n-2)](V+n)$. Therefore we can find the coordinates of comparable points in explicit form, and thereby convince ourselves that by a proper choice of n we can realize both cases.

We shall not stop to consider how the picture changes with a further increase in n. Let us consider the case where $n > 2\gamma/(\gamma - 1)$ (Fig.4). The asymptotic formula for $V \to \infty$ (4.1) shows that there exists a value of M_0 such that, for $M \geqslant M_0$, the solution of Cauchy's problem is continuous, unique and determined for all values of x and $t > 0$. Along with this it is obvious from Fig. 4 that for all $M < M_0$ the solution of Cauchy's problem is not continuous for all values of $t > 0$.

The case $n > 0$ and $\gamma > 3$ is analogous.

The final result can be formulated as follows. For $n > 0$ for any $\gamma > 1$ the solution of Cauchy's problem has the following possibilities:

(a) It can be continuous for all $t > 0$ but not all values of M. Here, for those values of M, for which the solution is continuous for all times $t > 0$, continuity can be achieved either in a unique way for a given M, or in an infinite number of ways.

(b) The solution is continuous for all values of M but here the continuity can be achieved in an infinite number of ways for each value of M.

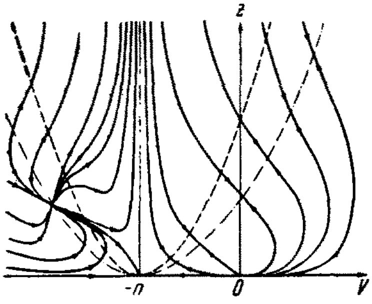

Fig. 4

Thus we can see that, for some initial conditions, a continuous solution of Cauchy's problem for the Kovaleskaya system of equations (1.1) either does not exist at all, inasmuch as an impassable limiting line appears in the x, t plane (of Ref. [7]), or else there exist infinitely many solutions for given initial conditions.

This fact, of the loss of uniqueness in the solution of Cauchy's problem for such systems with determined initial conditions, is worth taking note of, as to the best of the author's knowledge it has not appeared in the literature up to now.

Finally in the case $n = \infty$ the equations can be integrated in terms of elementary functions and the solution can be written in the following form:

$$\begin{aligned} u(x, t) &= M \sqrt{\frac{p_0}{\rho_0}} - \frac{p_0}{\rho_0}\frac{t}{a} \\ p(x, t) &= p_0 \exp\left(\frac{x}{a} - M \sqrt{\frac{p_0}{\rho_0}}\frac{t}{a} + \frac{1}{2}\frac{p_0 t^2}{\rho_0 a^2}\right) \\ \rho(x, t) &= \rho_0 \exp\left(\frac{x}{a} - M \sqrt{\frac{p_0}{\rho_0}}\frac{t}{a} + \frac{1}{2}\frac{p_0 t^2}{\rho_0 a^2}\right) \end{aligned} \tag{4.4}$$

Analysis of the asymptotic behavior of the solution of the system (1.13), (1.14) and (1.15) with $V \to \infty$ shows that, in the cases where the solution to Cauchy's problem exists and is determined for all x and $t > 0$, this solution at $t \to \infty$ has the form

$$u(x, t) \approx C_1(x) V \exp\left(-\frac{V}{C}\right), \quad p(x, t) \approx C_2(x) V^{\alpha} \exp\left(-n\frac{V}{C}\right)$$

$$\rho(x, t) \approx C_3(x) \exp\left[-(n-2)\frac{V}{C}\right], \quad t \approx C_4(x) \exp\left(\frac{V}{C}\right)$$

where $\alpha > 0$, C are constants. From here it follows that for $t \to \infty$,

$$u \to 0, \quad p \to 0 \ (n > 0), \quad p \to \infty \ (n < 0), \quad \rho \to 0 \ (n > 2)$$

$$\rho \to \infty \ (n < 2), \quad RT = \frac{p}{\rho} \to 0$$

(R = gas constant, T = absolute temperature).

The case of $n = \infty$ in this relation is exceptional, because from formula (4.4) it follows that at $t \to \infty$, $u \to \infty$; in this case $RT = RT_0 = p_0/\rho_0$ (the motion is isothermal). Note that solution (4.4) does not contain γ.

2. Piston Problem.

The problem of self-similar motion of gas that is created when the gas is displaced by a symmetrical piston is constructed as follows [5]. In a gas which is at rest and fills the whole space, and which has density ρ_0 and zero pressure, a sphere (cylinder or a flat layer in the cases of cylindrical or planar symmetry) is expanding according to the law $r_0 = ct^{n+1}/(n+1)$, where r_0 is the radius of the piston, t = time, c and n = constants. What is to be determined is the motion created in the ideal non-conducting perfect gas. This motion is described by some solution of system (1.1). It is self-similar, and the functions u, p, ρ have the form [5]:

$$u(r, t) = \frac{r}{t} V(\lambda), \quad p(r, t) = \rho_0 \frac{r^2}{t^2} P(\lambda), \quad \rho(r, t) = \rho_0 R(\lambda) \tag{5.1}$$

$$\lambda = \mu \frac{ct^{n+1}}{r} \qquad (\mu = \text{const})$$

The functions $V(\lambda)$, $P(\lambda) = \gamma^{-1} R(\lambda) Z(\lambda)$, $R(\lambda)$ are found in the ordinary equations (4), (5) and (6) of Ref. [5].

Let us replace t by $t + n\tau$ and let $\mu = 1/n$, $V = nV_1$, $P = n^2 P_1$, $R = R_1$, $c = (a/\tau)(n\tau)^{-n}$ and go over to the limit $n \to \infty$. In the result we will obtain for the motion of the piston the equation

$$r_0 = ae^{t/\tau} \tag{5.2}$$

and from equation (5.1) the following equations (indices omitted) (5.3)

$$u(r, t) = \frac{r}{\tau} V(\lambda), \quad p(r, t) = \rho_0 \frac{r^2}{\tau^2} P(\lambda), \quad \rho(r, t) = \rho_0 R(\lambda), \quad \lambda = \frac{a}{r} e^{t/\tau}$$

The differential equations then will become:

$$\frac{dZ}{dV} = Z \frac{[2 + \nu(\gamma - 1)] V (V-1)^2 - (\gamma - 1) V^2 (V - 1) - 2 [V - (\gamma-1)/\gamma] Z}{(V - 1) [V^2 (V - 1) - [2/\gamma + \nu V] Z]} \tag{5.4}$$

$$\frac{d \ln \lambda}{dV} = \frac{(V - 1)^2 - Z}{V^2 (V - 1) - (2/\gamma + \nu V) Z} \tag{5.5}$$

$$(V - 1) \frac{d \ln R}{d \ln \lambda} = \frac{V^2 (V - 1) - (2/\gamma + \nu V) Z}{Z - (V - 1)^2} + \nu V \tag{5.6}$$

Passage to the limit at the shock wave leads to the relations

$$V_2 - 1 = (V_1 - 1)\left[1 + \frac{2}{\gamma+1}\frac{Z_1 - (V_1-1)^2}{(V_1-1)^2}\right], \quad R_2 = R_1\frac{V_1 - 1}{V_2 - 1} \tag{5.7}$$
$$Z_2 = \left(\frac{\gamma-1}{\gamma+1}\right)^2 \frac{1}{(V_1-1)^2}\left[(V_1-1)^2 + \frac{2Z_1}{\gamma-1}\right]\left[\frac{2\gamma}{\gamma-1}(V_1-1)^2 - Z_1\right]$$

In this case the characteristic parabolas corresponding to (2.2) and (2.3) are

$$Z = (V-1)^2, \qquad Z = \frac{2\gamma}{\gamma-1}(V-1)^2 \tag{5.8}$$

The picture describing the possible transitions in the plane V, Z according to the relations (5.7) is identical to Fig. 1, if in the latter we substitute 1 in place of $-n$.

To construct the solution of the problem, let us consider the field of integral curves of equation (5.4), which is represented in Fig. 5. The arrows indicate the direction of the increase of the parameter λ along the integral curves.

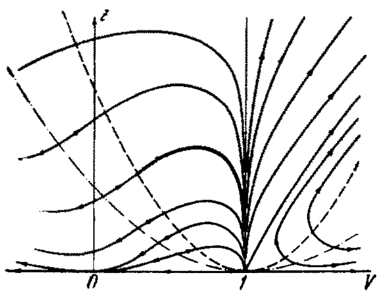

Fig. 5.

Evidently the solution of the problem should be discontinuous. The motion of the piston will create a strong shock wave which will propagate through the stationary gas and set it into motion. Corresponding to the state of rest (condition ahead of the shock wave) is the point $Z = V = 0$ in the V, Z plane. Corresponding to the state of gas behind the shock wave is the image of this point on the second of the parabolas (5.8) i.e. the point

$$V_2 = \frac{2}{\gamma+1}, \quad Z_1 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \tag{5.9}$$

The solution of the problem is given by the integral curve going from the point (5.9) to the point $Z = 0$, $V = 1$. In the vicinity of the latter point we have the following asymptotic representation:

$$Z \approx C_1(1-V)^{\frac{2}{2+\nu\gamma}}, \quad \lambda \approx C_2\left(V + \frac{2}{\nu\gamma}\right)^{\frac{1}{\nu}} \quad R \approx C_3\left(\frac{V + 2/\nu\gamma}{1-V}\right)^{\frac{2}{2+\nu\gamma}} \tag{5.10}$$

where C_1, C_2, C_3 are constants.

Satisfying the initial conditions on the piston, we establish that here $\lambda = 1$. This gives $C_2 = [\nu\gamma/(2+\nu\gamma)]^{1/\nu}$. Actual calculation of the solution of the problem can be by the method indicated in Ref. [5]. Namely, coming from point (5.9) we construct by means of numerical integration an integral curve, which in the vicinity of point $Z = 0$, $V = 1$ joins with the first of the asymptotic equations (5.10), and in this manner we can find the constant C_1. Further, we calculate the distribution of λ along the constructed integral curve by means of the second of formulae (5.10) and equation (5.5). Finally with the help of equation (5.6) and the condition at a strong shock wave $R_2 = (\gamma+1)/(\gamma-1)$, we calculate the distribution of R along the integral curve. The joining of this distribution with the third of the asymptotic formulas (5.10) gives the constant C_3. From this formula we can see that the density on the piston is equal to infinity, which is to be expected, as the solution constructed here is a limiting one for the regular similarity solutions with $n \to \infty$, and in Ref. [5] it is indicated that for $n > 0$ the density on the piston is always equal to infinity. In the constructed solution, as well as in the solutions of Ref. [5] for $n > 0$, the pressure on the piston is finite [c.f. formulas (5.10)] and the temperature there is equal to zero.

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